Metrical Combinatorics and the Real Half of the Fibonacci Sequence

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Languages with stress group syllables into metrical feet (Halle and Idsardi 1995, Hayes 1995)—non-exhaustive groups of contiguous syllables. The size of feet in natural languages ranges from unary (a single syllable) to unbounded (as many syllables as possible); in addition syllables can also remain unfooted. Under these conditions, the number of possible metrical footings for a string of *n* syllables is known to be Fib(2*n*) (Idsardi 2008), where Fib(*n*) is the Fibonacci sequence, as in (1)[1] where the n^{th} element is the sum of the previous two elements (for example, 13 = 8 + 5).

(1)	*n*:	0	1	2	3	4	5	6	7	…
	Fib(*n*):	$\underline{1}$	1	$\underline{2}$	3	$\underline{5}$	8	$\underline{13}$	21	…

For example, a string of two syllables (here notated with 'x's) can be non-exhaustively footed in five ways (= Fib(4)): (xx), (x)(x), (x)x, x(x), and xx. In contrast, if footing were required to be exhaustive (that is, if every syllable had to belong to some foot) then a string of two syllables could only be footed in two ways: (xx) and (x)(x). It is easy to see from the bracketed grid representations that the number of possible exhaustive footings of a string of *n* syllables must be $2^{n\text{-}1}$ as every exhaustive footing must begin and end with foot-boundaries and between each pair of x's we have a binary choice between having a foot juncture and not having one. Since there are two choices for each space between x's and there are *n*-1 spaces between *n* x's, it follows directly that there are $2^{n\text{-}1}$ distinct exhaustive footings.

As a consequence, only half of the Fibonacci numbers (those underlined in (1): 1, 2, 5, 13, …) are solutions to the task of creating non-exhaustive footings; the other half (3, 8, 21, …) are not. An intriguing question is: Why is it the one half of the sequence and not the other? We venture some speculations about potential answers.

In 1680, Cassini (1733) discovered a relation among successive members of the Fibonacci sequence, expressed in (2):[2]

(2) $\text{Fib}(n)^2 - \text{Fib}(n-1) \cdot \text{Fib}(n+1) = (-1)^n$

1 The Fibonacci sequence can also be defined to start with 0: 0, 1, 1, 2, 3, …
2 This relation was independently discovered by Simson (1753).

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That is, the square of any Fibonacci number is equal to the product of the two flanking Fibonacci numbers, give or take one. For example, $Fib(4)^2 - Fib(3) \cdot Fib(5) = 25 - 3 \cdot 8 = 1 = (-1)^4$ and $Fib(5)^2 - Fib(4) \cdot Fib(6) = 82 - 5 \cdot 13 = -1 = (-1)^5$. Rearranging (2) gives (3):

(3) $Fib(n)^2 - (-1)^n = Fib(n-1) \cdot Fib(n+1)$

The left-hand side of (3) has two possible expansions, depending on whether *n* is odd or even, as in (4):

(4) a. *n* is even: $Fib(n)^2 - 1$
b. *n* is odd: $Fib(n)^2 + 1$

We can now see that (4a) has the form $(x^2 - 1)$ and (4b) has the form $(x^2 + 1)$. Elementary algebraic polynomial factorization (Herstein 1977) shows that (4a) has real-valued roots, (5a), whereas (4b) only has complex-valued roots, (5b):

(5) a. $Fib(n)^2 - 1 = [Fib(n) - 1][Fib(n) + 1]$
b. $Fib(n)^2 + 1 = [Fib(n) - i][Fib(n) + i]$ (where $i^2 = -1$)

Thus, for example, $Fib(3) \cdot Fib(5) = 3 \cdot 8 = 24 = 4 \cdot 6 = [Fib(4) - 1][Fib(4) + 1]$. Only the even-numbered Fibonacci numbers (here, Fib(4)) show up in the real-valued roots, and this is the same Fibonacci subset that characterizes the number of valid metrical groupings of strings of *n* syllables.

In conclusion, the 'metrical' half of the Fibonacci sequence is also the 'real-valued' half of the sequence (in the sense of (5)). Evidently, the Fibonacci character of footing arises just when we allow for non-exhaustive footing, as exhaustive footings can be counted as a simple set of independent binary choices. Generally, the Fibonacci sequence is associated with a number of 'edge of chaos' effects, especially systems which illustrate *dynamical frustration* (Binder 2008); systems in which opposing forces cannot reach an equilibrium solution. We speculate that the 'forces' operative here in defining non-exhaustive footings could be the local coherence of syllables into feet clashing with word-level properties of footing. Another potential view of the emergent complexity observed here would be that sequences of footed syllables can be metrically distinct — for example, (x)(x) ≠ (xx) — whereas all sequences of unfooted syllables are the same; thus we have asymmetric growth patterns in the footed and unfooted portions of syllabic strings resulting in Fibonacci complexity.

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